Name : Khaireddine Dhahri.

Title : Deforming discontinuous actions for nilpotent Lie groups and some compact extentions.

Position : Researcher.

Date of defense : December, 2019.

Referees : K. Tounsi (Sfax) and D. Manchon (CNRS, Clermont-Ferrand).

Abstract : Geometry is a branch of mathematics that studies the figures of 3-dimension space (Euclidean geometry) and, since the $XVIII^{th}$ century, studies figures of other types of spaces (projective geometry, non-Euclidean geometry etc). Some methods of studying figures of these spaces have become autonomous branches of mathematics : topology, differential geometry, algebraic topology etc. It is therefore difficult to define today what concept of geometry is, in order to encompass all these geometries. In a strict way, geometry is "the study of shapes and sizes of figures". This definition is consistent with the emergence of geometry as a science under Greek civilisation during the classic epoch. According to Jean-Pierre Kahane's report, this definition coincides with the idea that geometry, as taught subject, is "the place where you learn to apprehend space". Questions raised during the XIX^{th} century have led to a reconsideration of the notion of forms and spaces, by removing the rigidity of Euclidean distances and the possibility of deforming continuously a surface without preserving the induced metric, for example to deform a sphere to an ellipsoid. Studying these deformations gave rise to the emergence of the topology. Thus, the objectives of study of geometry are as follows : sets and topological spaces, whose notion of proximity and continuity are defined together by the concept of neighborhood. According to some mathematicians, topology is a part and parcel of geometry, or even a fundamental branch. This classification can be questioned by other authors. Based on Felix-Klein point of view, analytic geometry "synthesized, in fact, two characters that were later dissociated : its fundamentally metric character and homogeneity". The first character is found in metric geometry, which studies the geometric properties of distances. The second character defines geometry as the study of invariants of group actions. Thus, the notion of group is placed at the closely of geometry, motivated by examples of physics. The study of structural deformations is

intimately linked to the study of deformations of groups.

The Lie groups theory, founded in $1870-1880$ by the norwegian mathematician Marius Sophus Lie, at first, was considered as a rather marginal part of mathematics. The Lie theory was next developed in parallel with the progress of algebra, topology and differential geometry. Therefore, it is now a very active branch of mathematics, having a strong impact on other areas.

In particular, several authors were interested in the problem of determination of the deformation space, of the action of a discontinuous group, acting a homogeneous space. This problem is related to the study of some geometric structures of closed surfaces, such as projective structures, hyperbolic structures (Teichmuller's space), complex structures etc. Concerning the framework of non-Riemannian spaces, the problem of describing deformation spaces was laid and treated (theoretically) by T. Kobayashi ([13] and [15]). Under certain circumstances, deformation spaces were explicitly determinated and some of their differential and topological characteristics were studied. Let G be a locally compact group and let H be a subgroup of G, the action of a subgroup $\Gamma \subset G$ on $X = G/H$ is said to be :

1) Proper if, for any compact $S \subset X$ the set $\Gamma_S = \{\gamma \in \Gamma : \ \gamma \cdot S \cap S \neq \emptyset\}$ is compact.

2) Free if, for any $x \in X$, the isotropic group Γ_x is trivial.

3) Properly discontinuous if, it is proper and Γ is discrete.

In this context, the subgroup Γ is said to be a discontinuous group for the homogeneous space X, if Γ is a discrete subgroup of G and Γ acts properly and freely on X. Let now Γ be a discontinuous group for the homogeneous space G/H, the deformation space, noted $\mathscr{T}(\Gamma, G, H)$, is defined as the orbit space of the action of G on the parameter space $\mathscr{R}(\Gamma, G, H)$, defined as follows :

$$g \cdot \varphi = g\varphi g^{-1}, g \in G, \varphi \in \mathscr{R}(\Gamma, G, H),$$

$$\mathscr{R}(\Gamma, G, H) := \left\{ \varphi \in \mathrm{Hom}(\Gamma, G) \;\middle|\; \begin{array}{l} \varphi \text{ is injective and} \\ \varphi(\Gamma) \text{ acts properly discontinuously} \\ \text{ on } G/H \end{array} \right\}.$$

and $\mathrm{Hom}(\Gamma, G)$ is the space of homomorphisms $\Gamma \to G$ endowed with the pointwise convergence topology.

The main scope of present thesis enters in to the forme above, it consists in studying some deformation problems of certain homogeneous spaces G/H, when the underlying group is nilpotent or a compact extension of a nilpotent group. The thesis is organized as follows :

Chapter 1 : "Preliminaries"

The first chapter is a reminder of some basic concepts that will be useful later, such as topological groups action and the deformation of the properly discontinuous action by defining the parameter and deformation spaces, Clifford-Klein forms, stability and rigidity and by treating some examples.

Chapter 2 : "Deforming discontinuous actions for Heisenberg motion groups"

In the second chapter, we study the existence of the Calabi-Markus phenomenon : given a necessary and sufficient condition for the existence of an infinite discrete subgroup of G which acts properly on G/H and the compact quotients of the homogenous spaces G/H by discrete subgroups Γ of G. It is said that the homogeneous space G/H admits a compact quotient, if there is a discrete subgroup Γ of G acting properly on G/H and as such the quotient $\Gamma\backslash G/H$ is compact. For example, any closed Riemann surface M with genus ≥ 2 is biholomorphic to a compact Clifford-Klein form of the Poincare plane $G/H = PSL(2,\mathbb{R})/SO(2)$. On the other hand, there is no compact Clifford-Klein form of the hyperboloid of one sheet $G/H = PSL(2,\mathbb{R})/SO(1,1)$. Even more, there is no infinite discrete subgroup of G which acts properly discontinuously on G/H (the Calabi-Markus phenomenon). So, not all discrete subgroup of G can act properly discontinuously on a homogeneous manifold G/H if H is not compact. More generally, Γ measures the defect of connectivity of the Clifford-Klein form $X = \Gamma\backslash G/H$ in the sense that if G/H is simply connected Γ is exactly the fundamental group of $X = \Gamma\backslash G/H$. Else, we show that the universal cover $\widetilde{X}$ of X is such that :

$$\widetilde{X}/\pi_1(X) = \Gamma\backslash G/H,$$

where $\pi_1(X)$ refers to the fundamental group of X.

Question 1. *Does G/H admit an infinite discontinuous group ?*

Question 2. *Does G/H admit a compact Clifford-Klein form ?*

Thus, to understand the local structure of the deformation space, we introduce the notion of local rigidity. An homomorphism $\varphi \in \mathscr{R}(\Gamma, G, H)$ is said to be locally rigid(resp.

rigid), if the orbit of φ by the inner conjugation is an open one in $\mathscr{R}(\Gamma, G, H)$ (resp. in $\mathrm{Hom}(\Gamma, G)$). This equivalently means that any point sufficiently close to φ must be combined with φ by an inner automorphism. A homomorphism locally rigid φ is isolated in the deformation space $\mathscr{T}(\Gamma, G, H)$. If any $\varphi \in \mathscr{R}(\Gamma, G, H)$ is locally rigid then, the deformation space is discrete.

In [20], Weil introduced the notion of the local rigidity of homomorphisms in the case where H is compact. In addition, for the symmetrical Riemannian spaces, Selberg's and Weil's have proven the following theorem :

Theorem 0.0.1. [*20*] *Let G be a simple linear non-compact Lie group and H its maximum compact subgroup. Then, the following statements are equivalent :*

1. *There exist an $\varphi : \Gamma \longrightarrow G$ such that $\varphi \in \mathscr{R}(\Gamma, G, H)$ is locally rigid.*
2. *G is locally isomorphic to $SL_2(\mathbb{R})$.*

In [11], Kobayashi proved that in the reductive case, local rigidity can fail even for irreducible symmetrical spaces of large dimensions. In [15], he has generalized the notion of local rigidity in the case where H is not compact. He showed that when G/H is non-Riemannian, where H is not compact, the local rigidity is not satisfied in general. In [13], he generalised Selberg-Weil theorem of local rigidity for non-Riemannian homogeneous spaces by showing the following theorem :

Theorem 0.0.2. *Let G be a simple linear non-compact linear Lie group, Γ a discret uniform subgroup of G and $(G', H') = (G \times G, \Delta_G)$ where Δ_G denotes the diagonal group of G. Then, the following statements are equivalent :*

1. *There exist an $\varphi : \Gamma \longrightarrow G$ such that $\varphi \times 1 \in \mathscr{R}(\Gamma, G', H')$ is not locally rigid.*
2. *G is locally isomorphic to $SO_n(n, 1)$ ou $SU(n, 1)$.*

In[1], Abdelmoula, Baklouti and Kedim proved a rigidity theorem, similar to that of Selberg, Weil and Kobayashi in the case of maximum exponential homogeneous spaces.

Theorem 0.0.3. *Let G be an exponential Lie group, H a non-normal connected maximum subgroup of G and a discontinuous subgroup Γ for the homogenous space G/H. Then, the following assertions are equivalent :*

1. *G is isomorphic to the $ax + b$.*

2. Any homomorphism in $\mathscr{R}(\Gamma, G, H)$ is locally rigid.

Question 1. *Can we characterize all locally rigid homomorphisms for a given triplet (G, H, Γ) ?*

The answer to this question as well as the explicit description of the parameter and deformation spaces have become very important in recent years, in particular, the case of the connected resolvable Lie groups. In [16], Kobayashi and Nasrin study the case where $\Gamma \simeq \mathbb{Z}^k$ act discontinuously on $G/H \simeq \mathbb{R}^{k+1}$ by affine transformations. In [2], Baklouti, Dhieb and Tounsi treated the case of the Heisenberg motion group $\mathbb{H}_{2n+1}$ and that of Baklouti, El Aloui and Kedim, recently been completed studies the case of the 2-step nilpotent Lie groups. In [7] and [8], Baklouti, Khlif and Koubaa treated the deformation of discontinuous groups acting on certain homogeneous spaces for the case of connected simply connected filiform Lie groups. The authors showed that the local rigidity is not satisfied. This work affirms the Baklouti's-conjecture [5] :

Conjecture : Let G be a nilpotent related Lie group, H a connected closed subgroup of G and Γ a discontinuous group for the homogeneous space G/H. Then, the parameter space admits a rigid point, if and only if it is finite.

Motivated by recent work on the study of deformations of discontinuous subgroups of Lie groups, the team of Ali Baklouti is currently tackling this subject. Answers are given for Euclidean motion groups and compact extension of $\mathbb{R}^n$. Remarkably, they appear to be divergent : the proved results cannot be canonized into a common formulation. Expect it to that, each structure has its own specificities. In this context, I have been suggested to study the case of the Heisenberg motion group.

Consider the Heisenberg group $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z, t)(w, s) = (z + w, t + s - \frac{1}{2}\mathrm{Im}\langle z, w\rangle).$$

The group $\mathbb{U}(n)$ of $n \times n$ complex unitary matrices acts on $\mathbb{H}_n$ by the automorphisms

$$A(z, t) = (Az, t) \text{ for each A} \in \mathbb{U}_n, z \in \mathbb{C}^n \text{ and } t \in \mathbb{R}.$$

The Heisenberg motion group is then the semi-direct product $G = \mathbb{U}(n) \ltimes \mathbb{H}_n$ with the multiplication law :

$$(A, z, t)(B, w, s) = (AB, z + Aw, t + s - \frac{1}{2}\mathrm{Im}\langle z, Aw\rangle).$$

We have then obtained the following results.

Theorem 0.0.4. *Let G be a Heisenberg motion group and H a closed subgroup of G. Then, G/H admits an infinite discontinuous group, if and only if H is conjugate to a subgroup of G^1 or for any $r > 0$, $H \cap (\mathbb{U}_n \times B(0, r) \times \mathbb{R})$ is compact.*

Theorem 0.0.5. *Let H be a closed subgroup of G, the homogenous space G/H admits a compact Clifford-Klein form, if and only if either H or G/H is compact.*

Theorem 0.0.6. *Let G be the Heisenberg motion group, H is a closed subgroup of G and Γ a discontinuous group for G/H. Then the followings are equivalent :*

1. *Any point of $\mathscr{R}(\Gamma, G, H)$ is strongly locally rigid.*
2. *$\mathscr{R}(\Gamma, G, H)$ admits a strongly locally rigid point.*
3. *$\mathscr{R}(\Gamma, G, H)$ admits a locally rigid point.*
4. *Γ is finite.*

In the last two chapters, we will be interested in nilpotent Lie groups. The description of the parameter and deformation spaces is a fundamental tool to study the geometrical and the topological properties of these spaces. The objective of this part of the thesis is to focus on the following questions :
1) Give an explicit description of the spaces $\text{Hom}(\Gamma, G)/G$, $\mathscr{R}(\Gamma, G, H)$ and $\mathscr{T}(\Gamma, G, H)$.
2) When is the deformation space a Hausdorff space ?
3) Stability problem : determine the set of stable parameters. A homomorphism $\varphi \in \mathscr{R}(\Gamma, G, H)$ is said to be stable, if there is an open set in $\text{Hom}(\Gamma, G)$, which contains φ and is contained in $\mathscr{R}(\Gamma, G, H)$.

Chapter 3 :"Deformation problems on nilpotent Lie groups"

The problem of the description of this space was raised by T. Kobayashi for non-Riemannian sets in [15] and [14]. The major difficulty in describing the parameter space is the characterization of the proper action of G on the homogeneous space G/H. This issue was addressed by several authors. One of the problems in studying the actions of a connected closed subgroup K of a Lie group G on the homogeneous space G/H, where H is also a connected closed subgroup of G, is the characterization of the proper action. In this context, the first question asked is “When does K act properly on G/H ?”.

So, Kobayashi introduced the notion of compact intersection, denoted (CI) and posed the following problem : "When does the property (CI) imply the proper action of K on G/H ?". In [12], an affirmative answer was given by Kobayashi in the case where (G, H, K) is reductive. However, this question may fail to hold in the case where G is reductive or abelian([11], [18]). On the other hand, Lipsman conjectured in [17] if G is a connected simply connected nilpotent Lie groups, H and K two closed subgroups of G, then the triplet (G, H, K) has the property (CI), if and only if the K-action on G/H is proper. In 2001, Nasrin argue that the conjecture is true for the 2-step nilpotent Lie groups [18]. However in 2004, Yoshino gave a counter-example for the conjecture by considering G is a 4-step nilpotent Lie group [19]. In the case where G is a completely solvable connected and simply connected, it is known that any closed subgroup of G admits a syndetic hull. This result leads to a topological identification (algebraic characterization) of the parameter and deformation spaces [9].

$$\mathscr{R}(\Gamma, G, H) \simeq \left\{ \varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} \dim(\varphi(\mathfrak{l})) = \dim \mathfrak{l} \\ \exp(\varphi(\mathfrak{l})) \text{ acts properly on } G/H \end{array} \right\}$$

where $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ are respectively the Lie algebras of G, H and L, the syndetic hull of Γ and

$$\mathscr{T}(\Gamma, G, H) \simeq \left\{ \varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} \dim(\varphi(\mathfrak{l})) = \dim \mathfrak{l} \\ \exp(\varphi(\mathfrak{l})) \text{ acts properly on } G/H \end{array} \right\} \Big/ G$$

where G acts on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ by $g \cdot \varphi = \mathrm{Ad}_g \circ \varphi$.

The study of proper and free actions on homogeneous spaces is a subject strongly linked to the deformation of the discontinuous groups acting on these spaces. This relationship appears first in the definition of discontinuous groups and also in the definition of space of deformation itself. In some circumstances of a connected simply connected, Lie group the deformation spaces were explicitly calculated and some of their characteristics differential and topological factors such as stability and rigidity were studied.

As an application of the general theory, T. Kobayashi and S. Nasrin studied in [16] the setup of a properly discontinuous action of a discrete subgroup $\Gamma \simeq \mathbb{Z}^k$ on $\mathbb{R}^{k+1} \simeq G/H$ through a certain two-step nilpotent affine transformation group G of dimension $2k + 1$,

when the connected subgroup H in question is $\mathbb{R}^k$. They explicitly determine the parameter and deformation spaces and characterize the set of non-stable morphisms. In [2] the authors studied the situation when G stands for the Heisenberg group and showed that the Hausdorff property of the deformation space is equivalent to the fact that $\mathscr{R}(\Gamma, G, H)$ is open in $\mathrm{Hom}(\Gamma, G)$ (which means that the stability property holds). In [7], A. Baklouti and F. Khlif studied the case of the connected and simply connected threadlike Lie groups and in [10], F. Khlif treated the case of the connected threadlike Lie groups. Beyond the nilpotent case, L. Abdelmoula, A. Baklouti and I. Kedim studied in [1] the situation where G is solvable exponential and H is maximal in G and A. Baklouti, I. Kedim and T. Yoshino [6] studied the situation, where G is an exponential Lie group and H contains $[G, G]$ or Γ is uniform in $[G, G]$. In [3], A. Baklouti, N. ElAloui and I. Kedim considered the setting where the underlying group G is two-step nilpotent. They proved first that if a pair (G, H) has the Lipsman property with G a connected simply connected nilpotent Lie group, then the parameter space is semi-algebraic. Besides, the authors provided an explicit description of the parameter and deformation spaces and established a stability and a Hausdorff theorems in the situation where G is two-step. In a more general case, A. Baklouti, M. Boussoffara and I. Kedim considered in [4] the case of three-step nilpotent Lie groups and generalized the results obtained in [3].
My work in this context is focused on the connected and simply connected nilpotent Lie groups. We provide a layering of Kobayashi's deformation space $\mathscr{T}(\Gamma, G, H)$ into Hausdorff spaces, which depends on the dimensions of G-adjoint orbits of the corresponding parameter space. Moreover, we give a sufficient condition on (Γ, G, H) such that the topology of the deformation space is Hausdorff.

Chapter 4 :"Stability of discontinuous groups acting on nilpotent homogeneous spaces"

The problem of stability is one of the most important problems in the deformation theory. It was introduced in [16] by Kobayashi-Nasrin and it may be one fundamental genesis to understand the local structure of the deformation space. Let G be a Lie group, H a connected closed subgroup of G and Γ a discontinuous group for the homogeneous space G/H.
In this chapter our objective is to give an algebraic description of the parametric space and give a stability theorem where the pair (G, H) has the Lipsman property.

Bibliographie

[1] Abdelmoula. L, Baklouti. A and Kédim. I. The Selberg-Weil-Kobayashi Rigidity Theorem For Exponential Lie Groups. Int Math Res Notices (17) : 4062-4084. doi : 10.1093/imrn/rnr172 (2012).

[2] Baklouti. A, Dhieb. S and Tounsi. K. When is the deformation space $\mathscr{T}(\Gamma, G, H)$ a smooth manifold, International Journal of Mathematics Vol. 22, No. 11 P. 1661-1681 (2011).

[3] Baklouti. A, ElAloui. N and Kédim. I. A rigidity Theorem and a Stability Theorem for two-step nilpotent Lie groups. J. Math. Sci. Univ. Tokyo. 19 (2012), 1-27.

[4] Baklouti. A, Boussoffara. M and Kédim. I. Some problems of deformations on three-step nilpotent Lie groups. Hiroshima mathematical journal. 49 (2018), 1-39.

[5] Baklouti. A. On discontinuous subgroups acting on solvable homogeneous spaces. Proc. Jap. Academy, 87, Serial A (2011) 173-177.

[6] A. Baklouti, I. Kédim and T. Yoshino. On the deformation space of Clifford-Klein forms of Heisenberg groups, Int. Math. Res. Not. (2008)16 (2008), doi :10.1093/imrn/rnn066.

[7] Baklouti. A and Khlif. F. Deforming discontinuous subgroups for threadlike homogeneous spaces, Geom. Dedicata. Vol. 146 117-140 (2010).

[8] Baklouti. A, Khlif. F. and Koubaa.On the geometry of stable discontinuous subgroups acting on Threadlike homogeneous spaces. Mathematical Notes, Springer. Vol 89, No. 6(2011)3-18.

[9] Baklouti. A and Kédim. I. On non-abelian discontinuous subgroups acting on exponential solvable homogeneous spaces, Int. Math. Res. Not. No. 7 1315-1345 (2010).

[10] Khlif. F. Rigidity of Discontinuous Groups for Threadlike Lie Groups, Graduate J. Math. 1 (2016), 43-53.

[11] Kobayashi. T. Proper action on homogeneous space of reductive type, Math. Ann. 285 (1989) 249-263.

[12] Kobayashi. T. Criterion of proper action on homogeneous space of reductive type, J. Lie theory 6, 147-163 (1996).

[13] Kobayashi. T. Deformation of compact Clifford-Klein forms of indefinite Riemannian homogeneous manifolds, Math. Ann. 310, 394-408 (1998).

[14] Kobayashi. T. Discontinuous groups for non-Riemannian homogeneous space, Mathematics Unlimited-2001 and Beyond, edited by B. Engquist and W. Schmid. Springer, 723-747 (2001).

[15] Kobayashi. T. On discontinuous groups on homogeneous space with non-compact isotropy subgroups, J. Geom. Phys. 12, 133-144 (1993).

[16] Kobayashi. T. And Nasrin S. Deformation of properly discontinuous action of $\mathbb{Z}^k$ on $\mathbb{R}^{k+1}$, Int. J. Math. 17, 1175-119 (2006).

[17] Lipsman. R. *Proper action and a compactness condition*, J. Lie theory 5 (1995), 25-39.

[18] Nasrin. S. Criterion of proper actions for 2-step nilpotent Lie groups. Tokyo J. Math. Vol. 24, No. 2, 535-543(2001)

[19] Yoshino T. Criterion of proper actions for 3-step nilpotent Lie groups, Internat. J. Math. 18, no 7, 783-795(2007)

[20] Weil A. Remarks on the cohomology of groups, Ann. Math. 80, 149-157(1964)